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More on the Calculation of the Integral

$$I_n(b) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos bx\, dx$$

By Henry E. Fettis

The evaluation of this integral has been the subject of two recent papers [1], [2]. Although the integral can be expressed in a simple analytical form, namely

$$(1)\qquad \left\{ I_n(b) = \frac{n}{2^{n-1}} \sum_{k=0}^{[(n-b)/2\}} \frac{(-)^k (n-b-2k)^{n-1}}{k!(n-k)!},\ b < n \right\},$$
$$= 0\,, \qquad\qquad b \geqq n\,,$$

(where $[(n-b)/2\}$ denotes the largest integer less than $(n-b)/2$), the use of the above expression for large n has not proved satisfactory. Alternative schemes in lieu of (1) have been proposed by Medhurst and Roberts [1] and Thompson [2]. These essentially are recursive-type methods, in which results for higher values of n and b are computed from starting values obtained for lower order and argument by the exact expression (1). Such schemes have the disadvantage that the direct computation for a given n and b is not possible. The present paper proposes a method which overcomes this difficulty and allows the integral to be computed directly. The formulae work equally well for small and large values of b, and are particularly well suited to computation for moderate and large n.

The basis of the present method is the Poisson summation formula [3]. In its most general form it may be written as follows

$$(2)\quad \sum_{k=-\infty}^{\infty} \exp[iku_1] f(a+kd) = \frac{1}{d}\sum_{m=-\infty}^{\infty} G\left(\frac{2\pi m + u_1}{d}\right) \exp[-i(a/d)(2\pi m + u_1)]$$

where G is the Fourier transform of f, namely

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$$G(y) = \int_{-\infty}^{\infty} f(x)e^{ixy}dx\,. \tag{3}$$

The formula (2) may be applied to the present problem by taking $a = 0$, $f(x) = (\sin x/x)^n$. We find that

$$G(y) = \pi I_n(y)\,. \tag{4}$$

Setting $u_1/d = t$, and noting that $f(x)$ is an even function, we get

$$\begin{aligned} &I_n(t) + I_n(t + 2\pi/d) + I_n(t - 2\pi/d) + I_n(t + 4\pi/d) + I_n(t - 4\pi/d) + \cdots + \\ &\quad = \frac{d}{\pi}\left[1 + 2\cos(dt)\left(\frac{\sin d}{d}\right)^n + 2\cos(2dt)\left(\frac{\sin 2d}{2d}\right)^n + \cdots +\right]. \end{aligned} \tag{5}$$

We now specialize Eq. (5) in two ways. First we set $t = 0$ and $d = 2\pi/b$, and obtain

$$\begin{aligned} &I_n(0) + 2I_n(b) + 2I_n(2b) + \cdots + \\ &\quad = \frac{2}{b}\left[1 + 2\left(\frac{\sin(2\pi/b)}{2\pi/b}\right)^n + 2\left(\frac{\sin(4\pi/b)}{4\pi/b}\right)^n + \cdots +\right]. \end{aligned} \tag{6}$$

Next, we set $t = b$ and $d = 2\pi/n$. This gives

$$\begin{aligned} &I_n(b) + I_n(n + b) + I_n(n - b) + I_n(2n + b) + I_n(2n - b) + \cdots + \\ &\frac{2}{\pi} = \left[1 + 2\cos\frac{2\pi b}{n}\left(\frac{\sin(2\pi/n)}{2\pi/n}\right)^n + 2\cos\frac{4\pi b}{n}\left(\frac{\sin(4\pi/n)}{4\pi/n}\right)^n + \cdots +\right]. \end{aligned} \tag{7}$$

In particular, if $b > n/2$, then $I_n(kb) = 0$ for $k > 1$, and Eq. (6) becomes

$$I_n(b) = \frac{1}{b}\left[1 + 2\left(\frac{\sin(2\pi/b)}{2\pi/b}\right)^n + 2\left(\frac{\sin(4\pi/b)}{4\pi/b}\right)^n + \cdots +\right] - \tfrac{1}{2} I_n(0)\,. \tag{8}$$

For $b = n$, we obtain Butler's result [4], namely

$$I_n(0) = \frac{2}{n}\left[1 + 2\left(\frac{\sin(2\pi/n)}{2\pi/n}\right)^n + 2\left(\frac{\sin(4\pi/n)}{4\pi/n}\right)^n + \cdots +\right]. \tag{9}$$

Next suppose $b < n/2$. Then $n - b > n/2$, and $nk \pm b > n(1 + k)/2 > n$ for $k > 1$. Hence (7) gives

$$\begin{aligned} I_n(b) + I_n(n - b) = \frac{2}{n}\Bigg[1 &+ 2\cos\left(\frac{2\pi b}{n}\right)\left(\frac{\sin(2\pi/n)}{2\pi/n}\right)^n + \cdots \\ &\cdots + 2\cos\left(\frac{4\pi b}{n}\right)\left(\frac{\sin(4\pi/n)}{4\pi/n}\right)^n + \cdots +\Bigg]. \end{aligned} \tag{10}$$

For $b = n/2$ both (8) and (10) give

$$I_n\left(\frac{n}{2}\right) = \frac{1}{n}\left[1 - 2\left(\frac{\sin(2\pi/n)}{2\pi/n}\right)^n + 2\left(\frac{\sin(4\pi/n)}{4\pi/n}\right)^n + \cdots +\right]. \tag{11}$$

Formulae (8) and (10) have the advantage that convergence is virtually unaffected by the value of "b". Further it is easy to predict in advance the number of terms needed for a prescribed accuracy by examining the magnitude of the successive maxima of $|\sin\theta/\theta|$. These maxima occur approximately at the points $\theta =$

$(2k + 1)(\pi/2)$ where $k = 1, 2, \cdots$. Their magnitude, for all practical purposes can be estimated as $1/(k + \frac{1}{2})$, and the contribution in the interval $[k\pi, (k + 1)\pi]$ as $2[k + \frac{1}{2}]^{-n}$, so that for $n > 10$ the summation can be terminated after three or less cycles of the integrand have been covered.

A formula similar to (6) can be obtained by taking $a = \pi/b$, $d = 2\pi/b$, $u = 0$ in Eq. (2).* This gives

$$(12) \qquad I_n(0) - 2I_n(b) + 2I_n(2b) - \cdots + = \frac{4}{b} \sum_{k=0}^{\infty} \left\{ \frac{\sin (k + \frac{1}{2})(2\pi/b)}{(k + \frac{1}{2})(2\pi/b)} \right\}^n .$$

For $b > n/2$ the left side reduces to $I_n(0) - 2I_n(b)$, so that

$$(13) \qquad I_n(b) = \tfrac{1}{2} I_n(0) - \frac{2}{b} \sum_{k=0}^{\infty} \left\{ \frac{\sin (k + \frac{1}{2})(2\pi/b)}{(k + \frac{1}{2})(2\pi/b)} \right\}^n .$$

Combining this with (8) and changing the index of summation, we obtain a formula for $I_n(b)$ which is free of $I_n(0)$:

$$(14) \qquad I_n(b) = \frac{1}{2b} \left[1 + 2 \sum_{m=1}^{\infty} (-)^m \left(\frac{\sin (m\pi/b)}{(m\pi/b)} \right)^n \right], \qquad b > n/2 ,$$

and by setting $b = n$ we obtain another expression for $I_n(0)$:

$$(15) \qquad I_n(0) = \frac{4}{b} \sum_{k=0}^{\infty} \left\{ \frac{\sin (k + \frac{1}{2})(2\pi/n)}{(k + \frac{1}{2})(2\pi/n)} \right\}^n .$$

Sample calculations of $I_n(b)$ for $n = 12$, $b = 0, 4, 6, 8$

k	$[\sin (2k\pi/n)/2k\pi/n]^n$		$[\sin (2k\pi/b)/2k\pi/b]^n$ $(b = 8)$		$\cos (2k\pi b/n)$	$\cos 2k\pi b/n[\sin (2k\pi/n)/2k\pi/n]^n$ $(b = 4)$	
1	.57498	50916	.28362	30021	−.5	−.28749	25458
2	.10233	49931	.00443	16094	−.5	−.05116	74966
3	.00443	16094	.00000	05337	1.0	.00443	16094
4	.00002	49841	.00000	00000	−.5	−.00001	24920
5	.00000	00024	.00000	00012	−.5	−.00000	00012
6	.00000	00000	.00000	00083	1.0	.00000	00000
7	.00000	00000			−.5	00000	00000
8	00000	00061			−.5	−.00000	00031
9	00000	00083			1.0	.00000	00083
10	00000	00004			−.5	−.00000	00002

$$.5 + \sum \left[\frac{\sin (2k\pi/n)}{2k\pi/n} \right]^n = 1.18177\ 66955 ; \qquad I_{12}(0) = .39392\ 55652$$

$$.5 + \sum (-)^k \left[\frac{\sin (2k\pi/n)}{2k\pi/n} \right]^n = -.47705\ 67278 ; \qquad I_{12}(6) = .00382\ 38787$$

$$.5 + \sum \left[\frac{\sin (2k\pi/b)}{2k\pi/b} \right]^n = .78805\ 51547 ; \qquad I_{12}(8) = .00005\ 10061$$

$$.5 + \sum \cos 2k\pi b/n \left[\frac{\sin (2k\pi/n)}{2k\pi/n} \right]^n = .16575\ 90791 ; \qquad I_{12}(4) = .05520\ 20202$$

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